CHAPTER 2
Functions: Understanding Rates of Change

Getting Started, p. 66

1. The slope between two points can be found by dividing the change in y by the change in x, $\frac{\Delta y}{\Delta x}$.

a) $\frac{7-3}{5-2} = \frac{4}{3}$

b) $\frac{5-(-1)}{-4-3} = -\frac{6}{7}$

2. a)
$$-1 - 1 = -2$$
$$-5 - (-1) = -4$$
$$-13 - (-5) = -8$$
$$-29 - (-13) = -16$$
$$-61 - (-29) = -32$$
Each successive first difference is 2 times the previous first difference. The function is exponential.

b) First differences
$$11 - 0 = 11$$
$$28 - 11 = 17$$
$$51 - 28 = 23$$
$$80 - 51 = 29$$
$$115 - 80 = 35$$
Second differences
$$17 - 11 = 6$$
$$23 - 17 = 6$$
$$29 - 23 = 6$$
$$35 - 29 = 6$$
The second differences are constant so the function is quadratic.

3. a) $0 = 2x^2 - x - 6$
$$0 = (2x + 3)(x - 2)$$
$$0 = 2x + 3 \text{ and } 0 = x - 2$$
$$0 - 3 = 2x + 3 - 3$$
$$-3 = 2x$$
$$-\frac{3}{2} = x$$
$$0 + 2 = x - 2 + 2$$
$$2 = x$$
The zeros are $-\frac{3}{2}$ and 2.

b) $0 = 2^x - 1$
$$0 + 1 = 2^x - 1 + 1$$
$$1 = 2^x$$
Any non-zero number raised to the exponent of 0 is 1, so $x = 0$.

c) $0 = \sin(x - 45°)$, $0° \le x \le 360°$
$\sin(0°)$, $\sin(180°)$, and $\sin(360°) = 0$.
$0° = x - 45°$, $180° = x - 45°$, and $360° = x - 45°$
$$0° + 45° = x - 45° + 45°$$
$$45° = x$$
$$180° + 45° = x - 45° + 45°$$
$$225° = x$$
$$360° + 45° = x - 45° + 45°$$
$$405° = x$$
Because $0° \le x \le 360°$, 405° cannot be a zero. The zeros are 45° and 225°.

d) $0 = 2\cos(x)$
$$0 = \cos(x)$$
For $-360° \le x \le 0°$, $\cos(-90°) = 0$ and $\cos(-270°) = 0$.
The zeros are $-90°$ and $-270°$.

4. a) $f(x)$ is compressed vertically by a factor of $\frac{1}{2}$.

b) $f(x)$ is stretched vertically by a factor of 2 and translated right 4 units.

c) $f(x)$ is stretched vertically by a factor of 3, reflected in the x-axis, and translated up 7 units.

d) $f(x)$ is stretched vertically by a factor of 5, translated right 3 units, and translated down 2 units.

5. a) \$1000 is P. 8% or 0.08 is i. $1 + i$ is 1.08. n is t. $A = P(1 + i)^n$ becomes $A = 1000(1.08)^t$

b) t is 3, $A = 1000(1.08)^3$ or \$1259.71

c) No, since the interest is compounded each year you earn more interest than the previous year. The interest earns interest.

6. a) $y = \sin x$ is a maximum at 90° so $15°t = 90°$ or $t = 6$.
$h(6) = 8 + 7\sin(15° \times 6)$. $h(6) = 15$ m. $y = \sin x$ is a minimum at 270° so $15°t = 270°$ or $t = 18$.
$h(18) = 8 + 7\sin(15° \times 18)$. $h(18) = 1$ m.

b) The period of $y = \sin x$ is 360°.
$15°t = 360°$ or $t = 24$ s.

c) $t = 30$. $h(30) = 8 + 7\sin(15° \times 30)$.
$h(30) = 15$ m.

7.

Linear relations	Nonlinear relations
constant; same as slope of line; positive for lines that slope up from left to right; negative for lines that slope down from left to right; 0 for horizontal lines.	variable; can be positive, negative, or 0 for different parts of the same relation

Rates of Change

Lesson 2.1 Determining Average Rate of Change, pp. 76–78

1. The average rate of change is equal to the change in y divided by the change in x.

a) $g(4) = 4(4)^2 - 5(4) + 1$
$= 64 - 20 + 1$
$= 45$
$g(2) = 4(2)^2 - 5(2) + 1$
$= 16 - 10 + 1$
$= 7$

Average rate of change $= \dfrac{45 - 7}{4 - 2} = 19$

b) $g(3) = 4(3)^2 - 5(3) + 1$
$= 36 - 15 + 1$
$= 22$
$g(2) = 7$

Average rate of change $= \dfrac{22 - 7}{3 - 2} = 15$

c) $g(2.5) = 4(2.5)^2 - 5(2.5) + 1$
$= 25 - 12.5 + 1$
$= 13.5$
$g(2) = 7$

Average rate of change $= \dfrac{13.5 - 7}{2.5 - 2} = 13$

d) $g(2.25) = 4(2.25)^2 - 5(2.25) + 1$
$= 20.25 - 11.25 + 1$
$= 10$
$g(2) = 7$

Average rate of change $= \dfrac{10 - 7}{2.25 - 2} = 12$

e) $g(2.1) = 4(2.1)^2 - 5(2.1) + 1$
$= 17.64 - 10.5 + 1$
$= 8.14$
$g(2) = 7$

Average rate of change $= \dfrac{8.14 - 7}{2.1 - 2} = 11.4$

f) $g(2.01) = 4(2.01)^2 - 5(2.01) + 1$
$= 16.1604 - 10.05 + 1$
$= 7.1104$
$g(2) = 7$

Average rate of change $= \dfrac{7.1104 - 7}{2.01 - 2} = 11.04$

2. a) i) According to the table, the height at $t = 2$ is 42.00 m and the height at $t = 1$ is 27.00 m.

$\dfrac{42 - 27}{2 - 1} = 15$ m/s

ii) According to the table, the height at $t = 4$ is 42.00 and $t = 3$ is 47.00 m.

$\dfrac{42 - 47}{4 - 3} = -5$ m/s

b) The flare is gaining height at 15 m/s and then loses height at 5 m/s.

3. $f(x)$ is always increasing at a constant rate. $g(x)$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, so the rate of change is not constant.

4. a) 1st half hour: $\dfrac{176 - 0}{0.5 - 0} = 352$ people/h

2nd half hour: $\dfrac{245 - 176}{1.0 - 0.5} = 138$ people/h

3rd half hour: $\dfrac{388 - 245}{1.5 - 1.0} = 286$ people/h

4th half hour: $\dfrac{402 - 388}{2.0 - 1.5} = 28$ people/h

5th half hour: $\dfrac{432 - 402}{2.5 - 2.0} = 60$ people/h

6th half hour: $\dfrac{415 - 432}{3.0 - 2.5} = -34$ people/h

b) the rate of growth of the crowd at the rally

c) A positive rate of growth indicates that people were arriving at the rally. A negative rate of growth indicates that people were leaving the rally.

5. a) Day 1: $\dfrac{203 - 0}{1 - 0} = 203$ km/day

Day 2: $\dfrac{396 - 203}{2 - 1} = 193$ km/day

Day 3: $\dfrac{561 - 396}{3 - 2} = 165$ km/day

Day 4: $\dfrac{739.5 - 561}{4 - 3} = 178.5$ km/day

Day 5: $\dfrac{958 - 739.5}{5 - 4} = 218.5$ km/day

Day 6: $\dfrac{1104 - 958}{6 - 5} = 146$ km/day

b) No; some days the distance travelled was greater than others.

6. The function is $f(x) = 4x$. To find the average rate of change find $\frac{\Delta f(x)}{\Delta x}$. The rate of change from $x = 2$ to $x = 6$ is:

$$\frac{f(6) - f(2)}{6 - 2} = \frac{4(6) - 4(2)}{6 - 2} = \frac{24 - 8}{6 - 2} = \frac{16}{4} = 4$$

The rate of change from 2 to 26 is:

$$\frac{f(26) - f(2)}{26 - 2} = \frac{4(26) - 4(2)}{26 - 2} = \frac{104 - 8}{26 - 2} = \frac{96}{24} = 4$$

The average rate of change is always 4 because the function is linear, with a slope of 4.

7. For any amount of time up to and including 250 minutes, the monthly charge is \$39, therefore the rate of change is 0 for that interval. After 250 minutes the rate of change is a constant 10 cents per minute. The rate is not constant.

8. a) Find the ordered pairs for the intervals given.
Interval i): (20, 20) and (0, 5)
Interval ii): (40, 80) and (20, 20)
Interval iii): (60, 320) and (40, 80)
Interval iv): (60, 320) and (0, 5)
Use this information to find the change in population over the change in time.

i) $\frac{20 - 5}{20 - 0} = \frac{15}{20} = \frac{3}{4}$ or 750 people per year

ii) $\frac{80 - 20}{40 - 20} = \frac{60}{20} = 3$ or 3000 people per year

iii) $\frac{320 - 80}{60 - 40} = \frac{240}{20} = 12$ or 12 000 people per year

iv) $\frac{320 - 5}{60 - 0} = \frac{315}{60} = 5.25$ or 5250 people per year

b) No; the rate of growth increases as the time increases.

c) Assume that the growth continues to follow this pattern and that the population will be 5 120 000 people in 2050.

9. The function is $h(t) = 18t - 0.8t^2$. The average rate of change is $\frac{\Delta h(t)}{\Delta t}$ for the interval $10 \leq t \leq 15$.

$$\frac{\Delta h(t)}{\Delta t} = \frac{h(15) - h(10)}{15 - 10} = \frac{18(15) - 0.8(15)^2 - (18(10) - 0.8(10)^2)}{15 - 10} = \frac{90 - 100}{15 - 10} = \frac{-10}{5} = -2 \text{ m/s}$$

10. a) The function is
$P(s) = -0.30s^2 + 3.5s + 11.5$
The average rate of change is $\frac{\Delta P(s)}{\Delta s}$.

i) $\frac{P(2) - P(1)}{2 - 1}$

$P(2) = -0.3(2)^2 + 3.5(2) + 11.15 = 16.95$

$P(1) = -0.3(1)^2 + 3.5(1) + 11.15 = 14.35$

$\frac{P(2) - P(1)}{2 - 1} = \frac{16.95 - 14.35}{2 - 1} = 2.6$

\$2.60 per sweatshirt

ii) $P(3) = -0.3(3)^2 + 3.5(3) + 11.15 = 18.95$

$P(2) = 16.95$

$\frac{P(3) - P(2)}{3 - 2} = \frac{18.95 - 16.95}{3 - 2} = 2$

\$2.00 per sweatshirt

iii) $P(4) = -0.3(4)^2 + 3.5(4) + 11.15 = 20.35$

$P(3) = 18.95$

$\frac{P(4) - P(3)}{4 - 3} = \frac{20.35 - 18.95}{4 - 3} = 1.4$

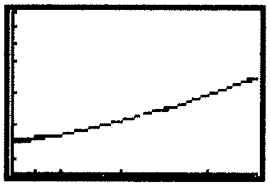
\$1.40 per sweatshirt

iv) $P(5) = -0.3(5)^2 + 3.5(5) + 11.15 = 21.15$

$\frac{P(5) - P(4)}{5 - 4} = \frac{21.15 - 20.35}{5 - 4} = 0.8$

\$0.80 per sweatshirt

b) The rate of change is still positive, but it is decreasing. This means that the profit is still increasing, but at a decreasing rate.
c) No; after 6000 sweatshirts are sold, the rate of change becomes negative. This means that the profit begins to decrease after 6000 sweatshirts are sold.

11. a) 

b) If we were to find the average rate of change of an interval that is farther in the future, such as 2025–2050 instead of 2010–2015, the average rate of change would be greater. The graph indicates that the change in population increases as time increases. The graph is getting steeper as the values of t increase.
c) The function is $P(t) = 50t^2 + 1000t + 20\,000$.
The average rate of change is $\frac{\Delta P(t)}{\Delta t}$.

i)
$$P(10) = 50(10)^2 + 1000(10) + 20\,000 = 35\,000$$
$$P(0) = 50(0)^2 + 1000(0) + 20\,000 = 20\,000$$
$$\frac{P(10) - P(0)}{10 - 0} = \frac{35\,000 - 20\,000}{10 - 0} = 1500 \text{ people per year}$$

ii)
$$P(12) = 50(12)^2 + 1000(12) + 20\,000 = 39\,200$$
$$P(2) = 50(2)^2 + 1000(2) + 20\,000 = 22\,200$$
$$\frac{P(12) - P(2)}{12 - 2} = \frac{39\,200 - 22\,200}{12 - 2} = 1700 \text{ people per year}$$

iii)
$$P(15) = 50(15)^2 + 1000(15) + 20\,000 = 46\,250$$
$$P(5) = 50(5)^2 + 1000(5) + 20\,000 = 26\,250$$
$$\frac{P(15) - P(5)}{15 - 5} = \frac{46\,250 - 26\,250}{15 - 5} = 2000 \text{ people per year}$$

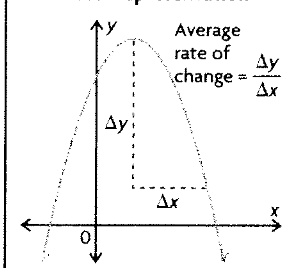
iv)
$$P(20) = 50(20)^2 + 1000(20) + 20\,000 = 60\,000$$
$$P(10) = 35\,000$$
$$\frac{P(20) - P(10)}{20 - 10} = \frac{60\,000 - 35\,000}{20 - 10} = 2500 \text{ people per year}$$

d) The prediction was correct.

12. Answers may vary. For example:
a) Someone might calculate the average increase in the price of gasoline over time. One might calculate the average decrease in the price of computers over time.
b) An average rate of change would be useful when there are several different rates of change over a specific interval.
c) The average rate of change is found by taking the change in y for the specified interval and dividing it by the change in x over that same interval.

13. The car's starting value is \$23 500. After 8 years the car is only worth \$8750.
The average rate of change in the value of the car is
$$\frac{8750 - 23\,500}{8 - 0} = \frac{-14\,750}{8} = -1843.75.$$
The value of the car decreases, on average, by \$1843.75 per year. As a percent of the car's original value, this is $\frac{1843.75}{23\,500} \times 100$, or 7.8% decrease, or -7.8%

14. Answers may vary. For example:

AVERAGE RATE OF CHANGE		
Definition in your own words	**Personal example**	**Visual representation**
the change in one quantity divided by the change in a related quantity	I record the number of miles I run each week versus the week number. Then, I can calculate the average rate of change in the distance I run over the course of weeks.	

15. Calculate the fuel economy for several values of x.

x	$F(x) = -0.005x^2 + 0.8x + 12$
10	19.5
20	26.0
30	31.5
40	36.0
50	39.5
60	42.0
70	43.5
80	44.0
90	43.5
100	42.0
110	39.5

The fuel economy increases as x increases to 80 and then decreases. The speed that gives the best fuel economy is 80 km/h.

Lesson 2.2 Estimating Instantaneous Rates of Change from Tables of Values and Equations, pp. 85–88

1. a) The function is $f(x) = 5x^2 - 7$.

The average rate of change is $\frac{\Delta f(x)}{\Delta x}$.

$$f(2) = 13, f(1) = -2, \frac{\Delta f(x)}{\Delta x} = 15$$

$$f(2) = 13, f(1.5) = 4.25, \frac{\Delta f(x)}{\Delta x} = 17.5$$

$$f(2) = 13, f(1.9) = 11.05, \frac{\Delta f(x)}{\Delta x} = 19.5$$

$$f(2) = 13, f(1.99) = 12.8, \frac{\Delta f(x)}{\Delta x} = 19.95$$

$$f(3) = 38, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 25$$

$$f(2.5) = 24.25, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 22.5$$

$$f(2.1) = 15.05, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 20.5$$

$$f(2.01) = 13.2, f(2) = 13, \frac{\Delta f(x)}{\Delta x} = 20.05$$

Preceding Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$1 \le x \le 2$	$13 - (-2) = 15$	$2 - 1 = 1$	15
$1.5 \le x \le 2$	8.75	0.5	17.5
$1.9 \le x \le 2$	1.95	0.1	19.5
$1.99 \le x \le 2$	0.1995	0.01	19.95

Following Interval	$\Delta f(x)$	Δx	Average Rate of Change, $\frac{\Delta f(x)}{\Delta x}$
$2 \le x \le 3$	$38 - 13 = 25$	$3 - 2 = 1$	25
$2 \le x \le 2.5$	11.25	0.5	22.5
$2 \le x \le 2.1$	2.05	0.1	20.5
$2 \le x \le 2.01$	0.2005	0.01	20.05

b) As the values of x get closer together on both sides of 2, the average rate of change gets closer to 20.

2. a) Find the average rate of change for intervals that approach 2.0 from both sides.

$$\frac{30.9 - 20.6}{2.0 - 1} = 10.3$$

$$\frac{30.9 - 26.98}{2.0 - 1.5} = 7.84$$

$$\frac{31.4 - 30.9}{3.0 - 2.0} = 0.5$$

$$\frac{32.38 - 30.9}{2.5 - 2.0} = 2.96$$

$$\frac{7.84 + 2.96}{2} = 5.4$$

$$\frac{10.3 + 0.5}{2} = 5.4$$

The instantaneous rate of change appears to be approaching 5.4.

b) Find the average rate of change for intervals that approach 2.0.

$$\frac{31.4 - 20.6}{3.0 - 1.0} = 5.4$$

$$\frac{32.38 - 26.98}{2.5 - 1.5} = 5.4$$

The instantaneous rate of change is approximately 5.4.

c) Answers may vary. For example: I prefer the centred interval method. Fewer calculations are required, and it takes into account points on each side of the given point in each calculation.

3. a) The population at 2.5 months is $P(2.5)$.

$$P(2.5) = 100 + 30(2.5) + 4(2.5)^2$$
$$= 200$$

b) $$P(0) = 100 + 30(0) + 4(0)^2$$
$$= 100$$

$$\frac{200 - 100}{2.5 - 0} = 40 \text{ raccoons per month}$$

c) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

$$f(2.51) = 100 + 30(2.51) + 4(2.51)^2$$
$$= 200.5004$$

$$\frac{200.5004 - 200}{0.01} = 50.04 \text{ or } 50 \text{ raccoons per month}$$

d) Part **a)** asks for the value of $P(t)$ at 2.5; part **b)** asks for the average rate of change over a certain interval; part **c)** ask for the instantaneous rate of change at 2.5—they are all different values.

4. Use the difference quotient to find the instantaneous rate of change.
$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

a) $f(-1.99) = 6(-1.99)^2 - 4$
$= 19.7606$
$f(-2) = 6(-2)^2 - 4$
$= 20$
$\frac{20 - 19.76}{-2 - (-1.99)} = -23.94$ or -24

b) $f(0.01) = 6(0.01)^2 - 4$
$= -3.9994$
$f(0) = 6(0)^2 - 4$
$= -4$
$\frac{-3.9994 - (-4)}{0.01} = 0.06$ or 0

c) $f(4.01) = 6(4.01)^2 - 4$
$= 92.4806$
$f(4) = 6(4)^2 - 4$
$= 92$
$\frac{92.48 - 92}{0.01} = 48.06$ or 48

d) $f(8.01) = 6(8.01)^2 - 4$
$= 380.9606$
$f(8) = 6(8)^2 - 4$
$= 380$
$\frac{380.96 - 380}{0.01} = 96.06$ or 96

5. Use the difference quotient to find the instantaneous rate of change.
$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.
The function is
$h(x) = -5x^2 + 3x + 65.$
$f(3.01) = -5(3.01)^2 + 3(3.01) + 65$
$= 28.7295$
$f(3) = -5(3)^2 + 3(3) + 65$
$= 29$
$\frac{28.7295 - 29}{0.01} = -27.05$ m/s or -27 m/s

6. Use the difference quotient to find the instantaneous rate of change.
$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.
The function is $H(t) = 125\,000(1.06)^t$.
$H(8.01) = 125\,000(1.06)^{8.01}$
$= 199\,347.13$
$H(8) = 125\,000(1.06)^8$
$= 199\,231.01$
$\frac{199\,347.13 - 199\,231.01}{0.01} = \$11\,612$ per year or about $11 610 per year

7. a) The function is $P(t) = -1.5t^2 + 36t + 6$.
The average rate of change is $\frac{\Delta y}{\Delta x}$.
$P(24) = -1.5(24)^2 + 36(24) + 6$
$= 6$
$P(0) = -1.5(0)^2 + 36(0) + 6$
$= 6$
$\frac{6 - 6}{24 - 0} = 0$ people/year

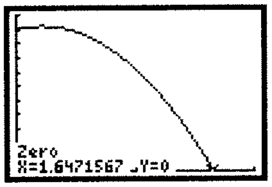
b) Answers may vary. For example: Yes, it makes sense. It means that the populations in 2000 and 2024 are the same, so their average rate of change is 0.

c) $P(12) = -1.5(12)^2 + 36(12) + 6$
$= 222$
$P(0) = 6$
$\frac{222 - 6}{12 - 0} = 18$ thousand/year
$P(24) = 6$
$P(12) = 222$
$\frac{6 - 222}{24 - 12} = -18$ thousand/year
The average rate of change during the first 12 years was 18 000 per year. During the second 12 years it was −18 000 per year. The population during year 0 is 6000 and during year 24 is 6000.

d) Because the average rate of change is the same on each side of 12, we know that the instantaneous rate of change would be 0 at 12.

8. Use the difference quotient to find the instantaneous rate of change.
$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.
The function is $V(t) = 18\,999(0.93)^t$.
$f(5.01) = 18\,999(0.93)^{5.01}$
$= 13\,207.79$
$f(5) = 18\,999(0.93)^5$
$= 13\,217.38$
$\frac{13\,207.79 - 13\,217.38}{0.01} = -959$
When the car turns five, it loses about $960/year.

9. a) The diver will hit the water when $h(t) = 0$.
$10 + 2t - 4.9t^2 = 0$
Use a graphing calculator to determine the value of t for which the equation is true.



The diver enters the water at about $t = 1.65$ s.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$h(1.66) = 10 + 2(1.66) - 4.9(1.66)^2 = -0.182\,44$$

$$h(1.65) = 10 + 2(1.65) - 4.9(1.65)^2 = -0.040\,25$$

$$\frac{-0.182\,44 + 0.040\,25}{0.01} = -14.219$$

The diver is travelling at a rate of about 14 m/s.

10. Use the centered interval method to estimate the instantaneous rate of change at $r = 5$. Find values of $V(r)$ on either side of 5.

$$V(5.1) = \frac{4}{3}\pi(5.1)^3 = 176.868\pi$$

$$V(4.9) = \frac{4}{3}\pi(4.9)^3 = 156.865\pi$$

$$\frac{176.868\pi - 156.865\pi}{5.1 - 4.9} = 314.21 \text{ or } 100\pi \text{ cm}^3/\text{cm}$$

Now, use the difference quotient to find the instantaneous rate of change.

$$V(5.01) = \frac{4}{3}\pi(5.01)^3 = 167.669\pi$$

$$V(5) = \frac{4}{3}\pi(5)^3 = 166.667\pi$$

$$\frac{167.669\pi - 166.667\pi}{0.01} = 314.63 \text{ or } 100\pi \text{ cm}^3/\text{cm}$$

11. David simply needs to keep track of the total distance that he's travelled and the amount of time that it has taken him to travel that distance. Dividing the distance travelled by the time required to travel that distance will give him his average speed.

12. a) Use a centered interval to find the instantaneous rate of change. $\frac{305 - 350}{5 - 3} = -22.5$ °F/min

b) Answers may vary. For example: A quadratic model for the oven temperature versus time is $y = -1.96x^2 - 9.82x + 400.71$. Using this model, the instantaneous rate of change at $x = 4$ is about -25.5 °F/min.

c) Answers may vary. For example, the first rate is using a larger interval to estimate the instantaneous rate.

d) Answers may vary. For example, the second estimate is better as it uses a much smaller interval to estimate the instantaneous rate.

13. Answers may vary. For example:

Method of Estimating Instantaneous Rate of Change	Advantage	Disadvantage
series of preceding intervals and following intervals	accounts for differences in the way that change occurs on either side of the given point	must do two sets of calculations
series of centred intervals	accounts for points on either side of the given interval in the same calculation	to get a precise answer, numbers involved will need to have several decimal places
difference quotient	more precise	calculations can be tedious or messy

14. a) The formula for finding the area of a circle is $A = \pi r^2$, where r is the radius. The average rate of change is $\frac{\Delta A}{\Delta r}$.

$$A = \pi(100)^2 = 10\,000\pi$$

$$A = \pi(0)^2 = 0$$

$$\frac{10\,000\pi - 0}{100 - 0} = 100\pi$$

The average rate of change is 100π cm²/cm.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$A = \pi(120.01)^2 = \pi(14\,402.4001)$$

$$A = \pi(120)^2 = 14\,400\pi$$

$$\frac{14\,402.4001\pi - 14\,400\pi}{0.01} \doteq 754.01 \text{ cm}^2/\text{cm or}$$

240π cm²/cm

15. The formula for the surface area of a cube given the length of a side is $V = 6s^2$, where s is the side length of the cube. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$SA = 6(3.01)^2 = 54.3606$$

$SA = 6(3)^2$
$= 54$

$$\frac{54.3606 - 54}{0.01} = 36.06 \text{ cm}^2/\text{cm}$$

The instantaneous rate of change is about 36 cm^2/cm.

16. The formula for finding the surface area of a sphere is $SA = 4\pi r^2$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a + h) - f(a)}{h}$, where h is a very small value.

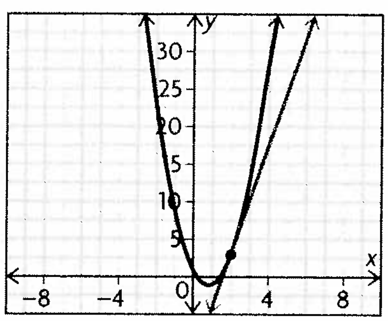
$SA = 4\pi(20.01)^2$
$= 1601.6004\pi$
$SA = 4\pi(20)^2$
$= 1600\pi$

$$\frac{1601.6004\pi - 1600\pi}{0.01} \doteq 502.78 \text{ cm}^2/\text{cm}$$

The instantaneous rate of change is about 502.78 cm^2/cm or 160π cm^2/cm.

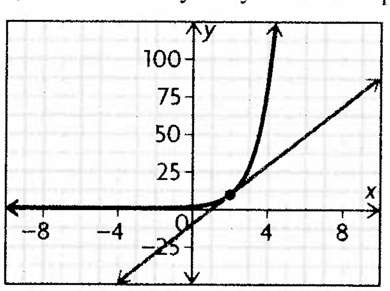
Lesson 2.3 Exploring Instantaneous Rates of Change Using Graphs, pp. 91–92

1. a) Answers may vary. For example:



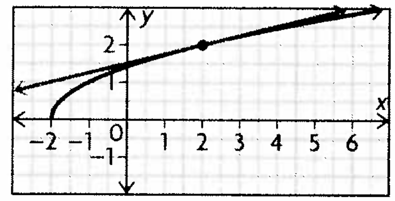
The slope is about 7.

b) Answers may vary. For example:



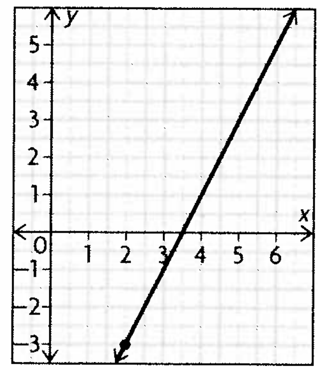
The slope is about 10.

c) Answers may vary. For example:



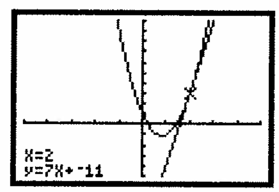
The slope is about 0.25.

d) Answers may vary. For example:

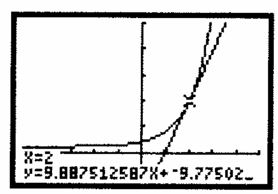


Because the graph is linear, the slope is the same everywhere. The slope is 2.

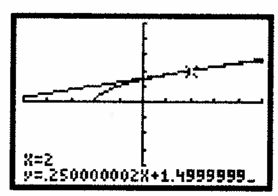
2. a)



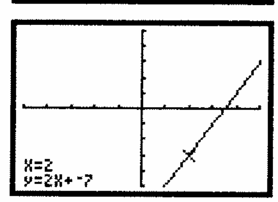
b)



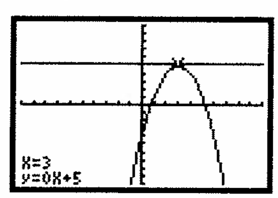
c)



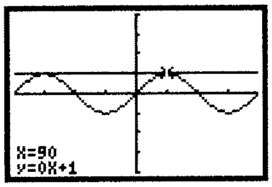
d)



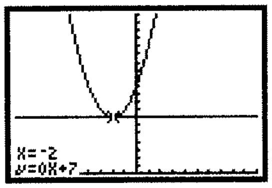
3. a) Set A:



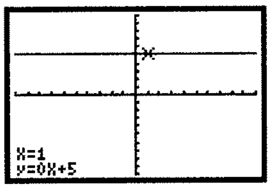
The slope of the tangent at $x = 3$ is 0.



The slope of the tangent at $x = 90°$ is 0.

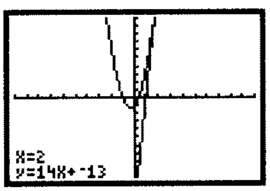


The slope of the tangent at $x = -2$ is 0.

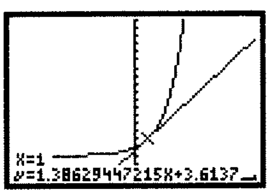


The slope of the tangent at $x = 1$ is 0.

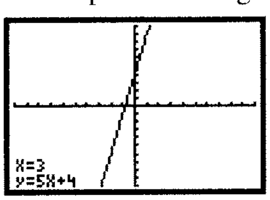
Set B:



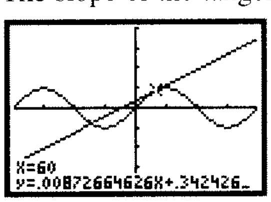
The slope of the tangent at $x = 2$ is 14.



The slope of the tangent at $x = 1$ is about 1.4.

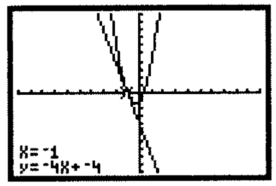


The slope of the tangent at $x = 3$ is 5.

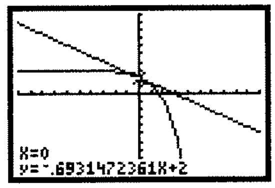


The slope of the tangent at $x = 60°$ is about 0.009.

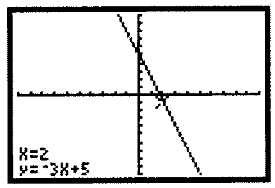
Set C:



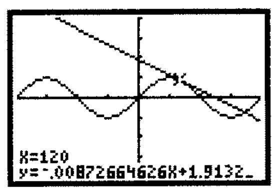
The slope of the tangent at $x = -1$ is -4.



The slope of the tangent at $x = 0$ is about -0.69.



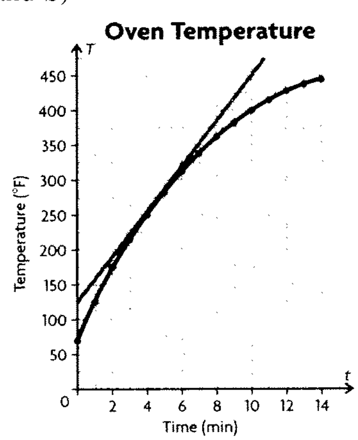
The slope of the tangent at $x = 2$ is -3.



The slope of the tangent at $x = 120°$ is about -0.009.

b) Set: A: All slopes are zero.
Set: B: All slopes are positive.
Set: C: All slopes are negative.

4. a) and **b)**



c) The y-intercept of the tangent line appears to be 125 °F. Find the slope between the points (0, 125) and (5, 280).

$$\frac{280 - 125}{5 - 0} = 31$$

The slope is 31.

d) Use the data points (6, 310) and (4, 250).

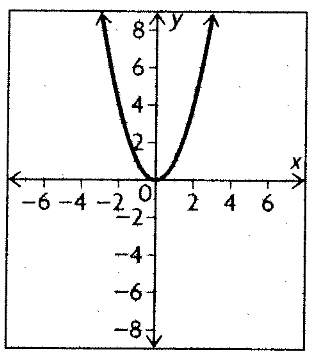
$$\frac{310 - 250}{6 - 4} = 30$$

The rate of change is about 30 °F/min at $x = 5$.

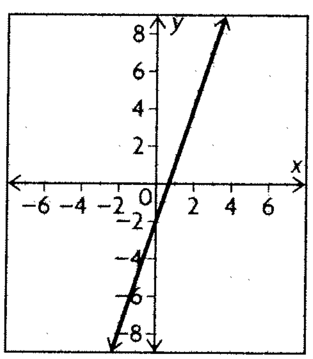
e) Answers may vary. For example: The answer in part d) is the slope of the line connecting two points on either side of $x = 5$. The answer in part c) is the slope of the line tangent to the function at point $x = 5$. The two lines are different and so their slopes will be different.

5. Answers may vary. For example, similarity: the calculation; difference: average rate of change is over an interval while instantaneous rate of change is at a point.

6. a)



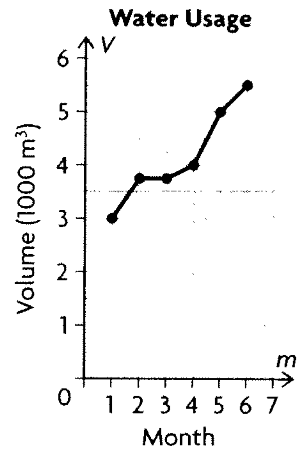
b)



c) From the graph, it appears that the tangent to the curve at (1.5, 2.25) would have the same slope as the secant line drawn.

Mid-Chapter Review, p. 95

1. a)



b) Rate of change is $\frac{\Delta f(x)}{\Delta x}$. Since we are looking for the amount of change between each month, Δx will always be 1 month. Therefore, we just need to find the difference in volume between each month.

m_1: $3.75 - 3.00 = 0.75$ 1000 m³/month or 750 m³/month

m_2: $3.75 - 3.75 = 0.00$ 1000 m³/month or 0 m³/month

m_3: $4.0 - 3.75 = 0.25$ 1000 m³/month or 250 m³/month

m_4: $5.10 - 4.00 = 1.10$ 1000 m³/month or 1100 m³/month

m_5: $5.50 - 5.10 = 0.40$ 1000 m³/month or 400 m³/month

c) Examine each of the answers from the previous exercises. The greatest amount is the greatest amount of change between two months.

$1.10 > 0.75 > 0.40 > 0.25 > 0.00$

The greatest amount of change occurred during $m4$, between April and May.

d) The change in y is the difference between the volume of water used in each month. The change in x is the difference between the numbers of the months.

$$\frac{5.50 - 3.75}{5 - 2} = 0.580 \times 1000 \text{ m}^3\text{/month or}$$

580 m³/month

2. a) The equation models exponential growth. This means that the average rate of change between consecutive years will always increase.

b) Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$f(10.01) = 1.2(1.05)^{10.01}$$
$$= 1.955\,627\,473$$
$$f(10) = 1.2(1.05)^{10}$$
$$= 1.954\,673\,552$$
$$\frac{1.955\,627\,473 - 1.954\,673\,552}{0.01} = 0.095\,39$$

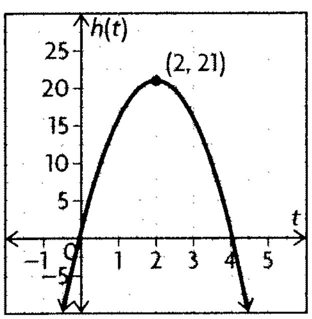
$0.095\,39 \times 10\,000 \doteq 950$ people per year

3. a) The average change for a specific interval is $\frac{\Delta h(t)}{\Delta t}$. The function is $h(t) = -5t^2 + 20t + 1$.

$$h(2) = -5(2)^2 + 20(2) + 1$$
$$= 21$$
$$h(0) = -5(0)^2 + 20(0) + 1$$
$$= 1$$
$$\frac{21-1}{2-0} = 10 \text{ m/s}$$
$$h(4) = -5(4)^2 + 20(4) + 1$$
$$= 1$$
$$h(2) = 21$$
$$\frac{1-21}{4-2} = -10 \text{ m/s}$$

b) $t = 2$; Answers may vary. For example: The graph has its vertex at (2, 21). It appears that a tangent line at this point would be horizontal.

$$\frac{(f(2.01) - f(1.99))}{0.02} \doteq 0$$

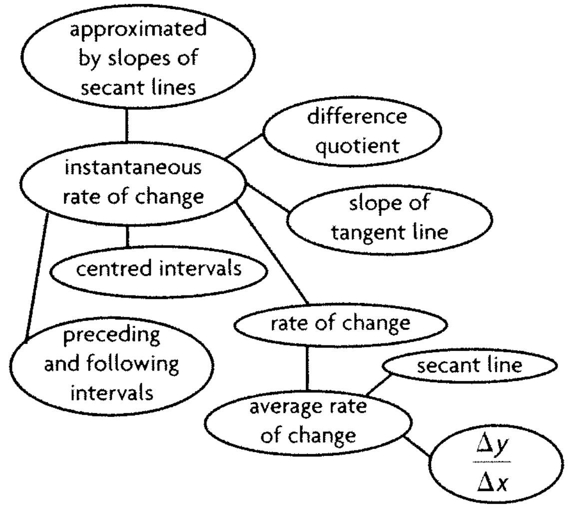


4. Use a centred interval.

$$d(20.01) = 0.01(20.01)^2 + 0.5(20.01)$$
$$= 14.009\,001$$
$$d(19.99) = 0.01(19.99)^2 + 0.5(19.99)$$
$$= 13.991\,001$$
$$\frac{d(20.01) - d(19.99)}{20.01 - 19.99} = 0.9.$$

So the instantaneous rate of change in the glacier's position after 20 days is about 0.9 m/day.

5. Answers may vary. For example:

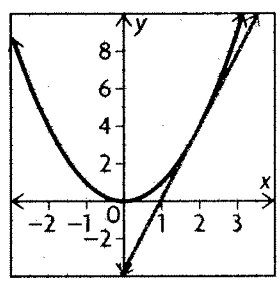


6. Answers may vary. For example: Find the value of y for different values of x on both sides of $x = 2$. Put this information in a table.

Points	Slope of Secant
(2, 9) and (1, 2)	7
(2, 9) and (1.5, 4.375)	9.25
(2, 9) and (1.9, 7.859)	11.41
(2, 9) and (2.1, 10.261)	12.61
(2, 9) and (2.5, 16.625)	15.25
(2, 9) and (3, 28)	19

The slope of the tangent line at (2, 9) is about 12.

7. Examine the graph.



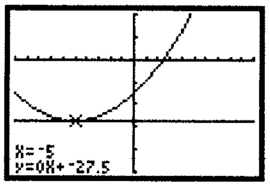
The tangent line appears to be passing through the points (1, 0) and (2, 4). Use this information to help determine the slope of the tangent line.

$$m = \frac{\Delta f(x)}{\Delta x}$$
$$= \frac{4-0}{2-1}$$
$$= 4$$

The slope of the line is 4.

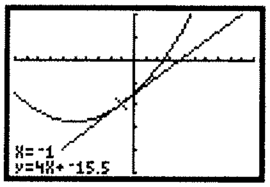
8. The instantaneous rate of change of the function whose graph is shown is 4 at $x = 2$.

9. a) Answers may vary. For example:



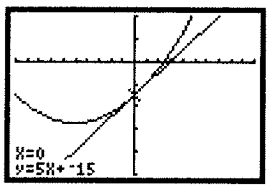
The slope is 0.

b)



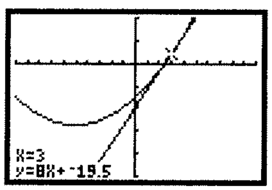
The slope is 4.

c)



The slope is 5.

d)



The slope is 8.

Lesson 2.4 Using Rates of Change to Create a Graphical Model, pp. 103–106

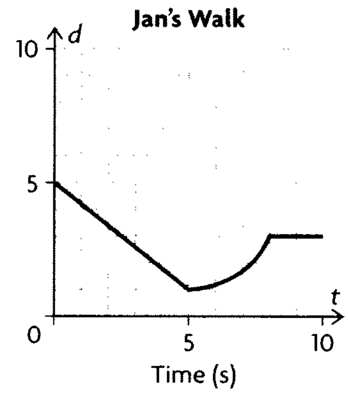
1. a) Graph a indicates that as time increases, distance also increases; C.

b) Graph b indicates that as time increases, distance decreases; A.

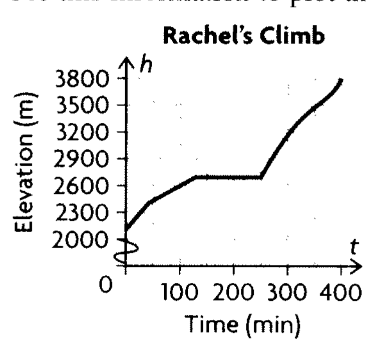
c) Graph c indicates that as time increases, the distance does not change; B.

2. Graph a indicates that distance is increasing at a steady rate over time, meaning that the speed is constant. However, graph b indicates that distance is decreasing at a steady rate over time—this also indicates that the speed is constant. Graph c indicates that distance does not change so speed is 0, a constant. All 3 are constant speed.

3. Draw a graph of Jan's distance from the sensor over time. Jan is 5 m away from the sensor, which means that her initial position is (0, 5). She then walks 4 m towards the sensor for 5 seconds, which means that she will be standing 1 m away from the sensor. Her second position will be (5, 1). She then walks 3 metres away for 3 seconds, which means that she will be 4 m away from the sensor. Her third position will be (8, 4). Jan then stops and waits for 2 seconds, which means she stays 4 m away from the sensor for 2 seconds. Her fourth position will be (10, 4). Use this information to draw the graph.



4. a) Answers may vary. For example, draw a graph of Rachel's distance over time while climbing Mt. Fuji. Rachel begins the climb at Level 5 and so her initial position is (0, 2100). She walks for 40 minutes at a constant rate to move from Level 5 to Level 6, which means that her second position will be (40, 2400). It then takes he 90 minutes to move from Level 6 to Level 7, which means that her third position will be (130, 2700). Rachel then decides to rest for 2 hours, which means that her position does not change. So her fourth position is (250, 2700). After her break, it took Rachel 40 minutes to reach Level 8. Her fifth position is (290, 3100). It took Rachel 45 minutes to go from Level 8 to Level 9. Her next position is (335, 3400). After the walk from Level 9 to Level 10, Rachel reached the top. This position can be represented as (395, 3740).
Use this information to plot the graph.



b) Use the data points from the previous question to determine Rachel's average speed during each part of her journey.

$$\frac{2400 - 2100}{40 - 0} = 7.5\text{ m/min}$$

$$\frac{2700 - 2400}{130 - 40} = 3.3\text{ m/min}$$

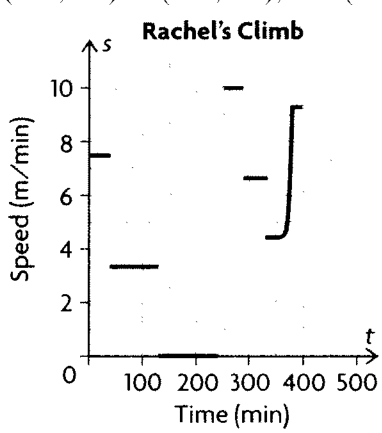
$$\frac{2700 - 2700}{250 - 130} = 0\text{ m/min}$$

$$\frac{3100 - 2700}{290 - 250} = 10.0\text{ m/min}$$

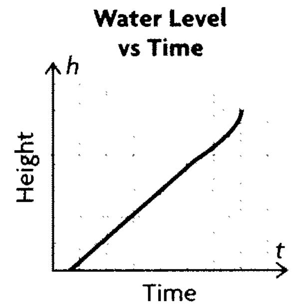
$$\frac{3400 - 3100}{335 - 290} = 6.7\text{ m/min}$$

$$\frac{3740 - 3400}{395 - 335} = 5.7\text{ m/min}$$

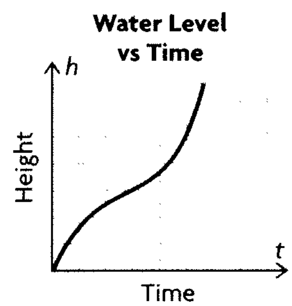
c) Answers may vary. For example, use Rachel's average rates to make a graph of her speed. During the first 40 minutes of her journey, her speed was 7.5 m/min. This can be represented by a straight line from (0, 7.5) to (40, 7.5). Rachel's speed during the next 90 minutes is 3.3 m/min. This speed can be represented by a straight line from (40, 3.3) to (130, 3.3). Rachel then rested for 2 hours. This can be represented with a straight line from (130, 0) to (250, 0). Rachel travelled at a rate of 10.0 m/min for the next 40 minutes. This speed can be represented by a straight line from (250, 10.0) to (290, 10.0). Then she travelled at a rate of 6.7 m/min for 45 minutes, 4.4 m/min for 45 minutes, and 9.3 m/min for 15 minutes. The speeds for these parts or her walk can be represented by the following segments: (290, 6.7) to (335, 6.7), (335, 4.4) to (380, 4.4), and (380, 9.3) to (395, 9.3).



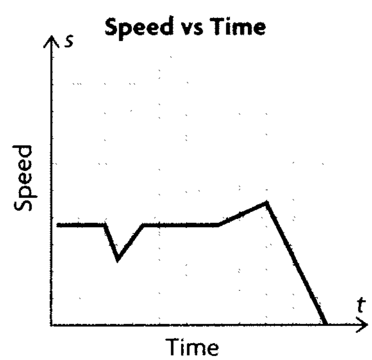
5. a) Answers may vary. For example, the 2 L plastic pop bottle has a uniform shape for the most part. Therefore, as long as the rate of water flowing into the bottle remains constant, the rate at which the height is changing will also remain constant.



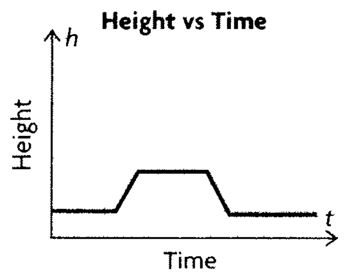
b) Answers may vary. For example, the circumference of the vase changes for any given height on the vase. Therefore, the rate of change of the height of the water flowing into the vase will vary over time—faster at the very bottom of the vase, slower in the middle and then faster again at the top.



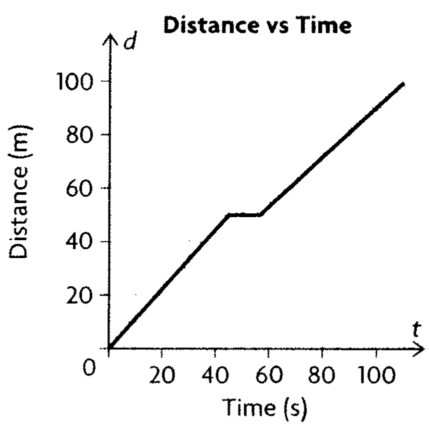
6. a) Answers may vary. For example, on a graph that represents John's speed, a constant speed would be represented by a straight line, any increase in rate would be represented by a slanted line pointing up, and any decrease in rate would be represented by a slanted line pointing down. John's speed over his bike ride could be represented following graph.



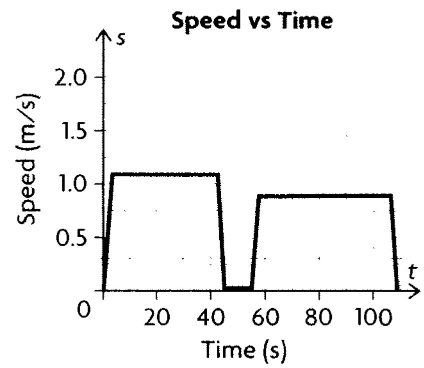
b) Answers may vary. For example, the first part of John's bicycle ride is along a flat road. His height over this time would be constant. As he travels up the hill, his height would increase. At the top of the hill, his height would again be constant. As he goes down the hill, his height would decrease. As he climbs the second hill his height would again increase. The graph of his height over time would look something like this.



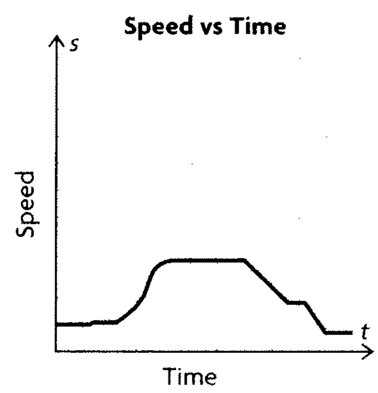
7. a) Kommy travels 50 m in 45 seconds. This means that his speed would be $\frac{50}{45} = 1.11$ m/s.
b) During the second part of his swim he travelled 50 m in 55 s. This means that his speed would be $\frac{50}{55} = 0.91$ m/s.
c) The graph of the first length would be steeper, indicating a quicker speed. The graph of the second length would be less steep, indicating a slower speed.
d) Answers may vary. For example: Use the information for part c) to draw the graph of Kommy's distance over time.



e) At $t = 50$, Kommy is resting, and so his speed would be 0.
f) Answers may vary. For example: Kommy's speed for the first 45 seconds is 1.11 m/s. This would be represented by the line segment from (0, 1.11) to (45, 1.11). Kommy then rests for 10 s, when his speed would be 0. This would be represented by a line segment from (45, 0) to (55, 0). Kommy's speed during the second half of his swim is 0.91 m/s. This would be represented by a line segment from (55, 0.91) to (110, 0.91).



8. a) A – if the rate at which a speed is increasing increases, this would be represented by an upward curve.
b) C – if the rate at which a speed is decreasing decreases over time, this would be represented by a curve that drops sharply at first and then drops more gradually.
c) D – if the rate at which a speed is decreasing increases, this would be represented by a downward curve.
d) B – if the rate at which a speed is increasing decreases, this would be represented by a curve that rises sharply at first and then rises more gradually.
9. Answers may vary. For example: Because the jockey is changing the horse's speed at a non-constant rate—at first slowly and then more quickly—the lines will have an upward curve when the horse is accelerating and a downward curve when decelerating. The horse's speed during the first part of the warm up is constant, which would be represented by a straight line. She then increases the horse's speed to a canter and keeps this rate for a while. Draw a graph of this information with speed over time.



10. a) Graph i) shows that distance is decreasing and then increasing. The first graph shows a person standing 5 m away from the motion sensor then moving to 2 m away. The person then moves back to 5 m away from the motion sensor. The person is always moving at a constant rate. Graph ii) shows a person's initial position being 6 m away from the motion sensor. This person then moves 2 m closer to the sensor over 2 seconds. Then, he or she rests for a second and then moves 2 m closer to the sensor over 2 more seconds. Finally, this person moves 2 m away from the sensor over 1 second to end up at a final position of about 4 m away from the sensor. The person is always moving at a constant rate.

b) For each graph, determine the (t, d) point for each position.

Graph A

$(0, 5)$; $(3, 2)$; $(6, 5)$

Graph B

$(0, 6)$; $(2, 4)$; $(3, 4)$; $(5, 2)$; $(6, 3.5)$

Use these points to find the various speeds.

Graph A

$\frac{2-5}{3-0} = -1$, so the speed is 1 m/s

$\frac{5-2}{6-3} = 1$, so the speed is 1 m/s

Graph B

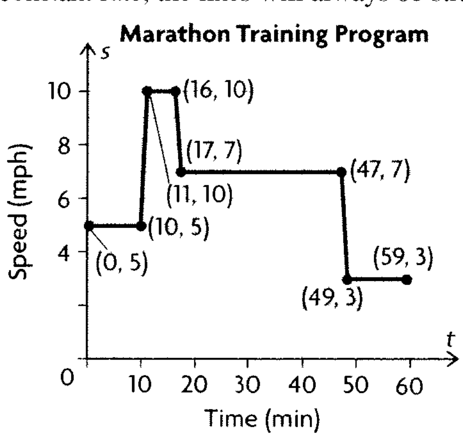
$\frac{4-6}{2-0} = -2$, so the speed is 2 m/s

$\frac{4-4}{3-2} = 0$, so the speed is 0 m/s

$\frac{2-4}{5-3} = -1$, so the speed is 1 m/s

$\frac{3.5-2}{6-5} = 1.5$, so the speed is 1.5 m/s

11. a) Answers may vary. For example: Draw a graph of the runner's speed over time. The runner's positions on the graph will be represented by the following points: $(0, 5)$, $(10, 5)$, $(11, 10)$, $(16, 10)$, $(17, 7)$, $(47, 7)$, $(49, 3)$, $(59, 3)$. Plot the points on a graph. Because the runner accelerates and decelerates at a constant rate, the lines will always be straight.



b) Use the data points on either side of $t = 10.5$ to estimate the instantaneous rate of change at that point. The points are $(10, 5)$, $(11, 10)$.

$\frac{10-5}{11-10} = 5$ mi/h/min

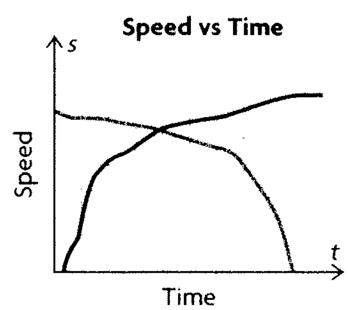
c) The runner's speed at minute 11 is 10 miles per hour. The runner's speed at minute 49 is 3 miles per hour.

$\frac{3-10}{49-11} = \frac{-7}{38} = -0.1842$ miles per hour per minute

d) The answer to part c) is an average rate of change over a long period, but the runner does not slow down at a constant rate during this period.

12. Answers may vary. For example: Walk from $(0, 0)$ to $(5, 5)$ and stop for 5 s. Then run to $(15, 30)$. Continue walking to $(25, 5)$ and end at $(25, 0)$. What is the maximum speed and the minimum speed on any interval? Create the speed time graph from these data.

13. Answers may vary. For example: Graphing both women's speeds on the same graph would mean that there are two lines on the graph. The first woman is decelerating; this means that her line would have a downward direction. Because she is decelerating slowly first and then more quickly, the line would also have a downward curve. The second woman is accelerating; this means that her line will have an upward direction. Because she is accelerating quickly at first and then more slowly, the graph would have a sharp upward curve. The line on the graph would look something like this:



14. If the original graph showed an increase in rate, it would mean that the distance travelled during each successive unit of time would be greater—meaning a graph that curves upward. If the original graph showed a straight, horizontal line, then it would mean that the distance travelled during each successive unit of time would be greater—meaning a steady increasing straight line on the second graph. If the original graph showed a decrease in rate, it would mean that the distance travelled during each successive unit of time would be less—meaning a line that curves down.

Lesson 2.5 Solving Problems Involving Rates of Change, pp. 111–113

1. Answers may vary. For example: Verify that the most economical production level occurs when 1500 items are produced by examining the rate of change at $x = 1500$. Because x is in thousands, use $a = 1.5$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$C(1.5) = 0.3(1.5)^2 - 0.9(1.5) + 1.675 = 1$$
$$C(1.501) = 0.3(1.501)^2 - 0.9(1.501) + 1.675 = 1.000\,000\,3$$
$$\frac{1.000\,000\,3 - 1}{0.01} = 0.000\,03$$

When 1500 items are produced, the instantaneous rate of change is zero. Therefore, the most economical production level occurs when 1500 items are produced.

2. The function is $P(t) = -20\cos(300°t) + 100$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

$$P(3) = -20\cos(300° \times 3) + 100 = 120$$
$$P(3.01) = -20\cos(300° \times 3.001) + 100 = 119.999\,73$$
$$\frac{119.999\,73 - 120}{0.001} = -0.27 \text{ or } 0$$

The blood pressure is dropping at a rate of 0 millimetres of mercury per second.

3. a) If $(a, f(a))$ is a maximum, then the points to the left of, and very close to the maximum, have a positive rate of change. As $x(a)$ approaches $(a, f(a))$ from the left, $y(f(a))$ is increasing because $(a, f(a))$ is a maximum.

b) If $(a, f(a))$ is a maximum, then the points to the right of, and very close to the maximum, have negative rate of change. As $x(a)$ moves away from $(a, f(a))$ to the right, $y(f(a))$ is decreasing because $(a, f(a))$ is a maximum.

4. a) If $(a, f(a))$ is a minimum, then the points to the left of, and very close to the maximum, have negative rate of change. As $x(a)$ moves toward $(a, f(a))$ from the left, $y(f(a))$ is decreasing because $(a, f(a))$ is a minimum.

b) If $(a, f(a))$ is a minimum, then the points to the right of, and very close to the maximum, have a positive rate of change. As $x(a)$ moves away from $(a, f(a))$ towards the right, $y(f(a))$ is increasing because $(a, f(a))$ is a maximum.

5. a) The leading coefficient is positive, and so the value given will be a minimum. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

Find $f(-6)$ and $f(-5.99)$. The function is $f(x) = 0.5x^2 + 6x + 7.5$.

$$f(-6) = 0.5(-6)^2 + 6(-6) + 7.5 = -10.5$$
$$f(-5.99) = 0.5(-5.99)^2 + 6(-5.99) + 7.5 = -10.499\,995$$
$$\frac{-10.5 - (-10.499\,995)}{0.01} = -0.0005 \text{ or } 0$$

The slope is very small, pretty close to zero, and so it can be assumed that $(-6, -10.5)$ is the minimum.

b) The leading coefficient is negative, and so the value given will be a maximum. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

Find $f(0.5)$ and $f(0.501)$. The function is $f(x) = -6x^2 + 6x + 9$.

$$f(0.501) = -6(0.501)^2 + 6(0.501) + 9 = 10.499\,994$$
$$f(0.5) = -6(0.5)^2 + 6(0.5) + 9 = 10.5$$
$$\frac{10.499\,994 - 10.5}{0.01} = -0.000\,6 \text{ or } 0$$

The number is very close to zero, and so we can assume that the point has an instantaneous rate of change of zero and is a maximum.

c) The function is $f(x) = 5\sin(x)$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h)-f(a)}{h}$, where h is a very small value.

Find $f(90°)$ and $f(90.01°)$.

$$f(90.01°) = 5\sin(90.01°) = 4.999\,999$$
$$f(90°) = 5\sin(90°) = 5$$
$$\frac{4.999\,999 - 5}{0.01} = -0.0001 \text{ or } 0$$

The number is very close to zero, and so we can assume that the instantaneous rate of change at the point is zero, and so the point is a maximum.

d) The function is $f(x) = -4.5\cos(2x)$. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

Find $f(0)$ and $f(0.01)$.

$$f(0.01) = -4.5\cos(2 \times 0.01)^\circ = -4.499\,999$$
$$f(0) = -4.5\cos(2 \times 0)^\circ = -4.5$$
$$\frac{-4.499\,999 - (-4.5)}{0.01} = 0.0001 \text{ or } 0$$

The number is very close to zero, and so we can assume that the instantaneous rate of change at the point is zero, and so the point is a maximum.

6. Examine the instantaneous rates of change on either side of the point in question. If the point to the left of the point in question is negative, then the point is a minimum. If the point to the left of the point in question is positive, then the point is a maximum. If the point to the right of the point in question is positive, then the point is a minimum. If the point to the right of the point in question is negative, then the point is a maximum. Use the difference quotient to find the instantaneous rate of change.

$\frac{f(a+h) - f(a)}{h}$, where h is a very small value.

a) $f(x) = x^2 - 4x + 5$; $(2, 1)$

Examine $x = 1$, which is to the left of $(2, 1)$.

$$f(1.01) = (1.01)^2 - 4(1.01) + 5 = 1.97$$
$$f(1) = (1)^2 - 4(1) + 5 = 2$$
$$\frac{1.97 - 2}{0.01} = -3$$

The instantaneous rate of change of $(1, 2)$ is negative and so $(2, 1)$ is a minimum.

b) $f(x) = -x^2 - 12x + 5.75$; $(-6, 41.75)$

Examine $x = -5$, which is to the right of $(-6, 41.75)$.

$$f(-4.99) = -(-4.99)^2 - 12(-4.99) + 5.75 = 40.7299$$
$$f(-5) = -(-5)^2 - 12(-5) + 5.75 = 40.75$$
$$\frac{40.7299 - 40.75}{0.01} = -2.01$$

The instantaneous rate of change of $(-5, 40.75)$ is -2.01, and so $(-6, 41.75)$ is a maximum.

c) $f(x) = x^2 - 9x$; $(4.5, -20.25)$

Examine $x = 5$, which is to the right of $(4.5, -20.25)$.

$$f(5.01) = (5.01)^2 - 9(5.01) = -19.899$$
$$f(5) = (5)^2 - 9(5) = -20$$
$$\frac{-19.899 - (-20)}{0.01} = 10.1$$

The instantaneous rate of change at $(5, -20)$ is positive and so $(4.5, -20.25)$ is a minimum.

d) $f(x) = 3\cos x$; $(0^\circ, 3)$

Examine $x = -1^\circ$, which is to the left of $(0^\circ, 3)$.

$$f(-0.99^\circ) = 3\cos(-0.99^\circ) = 2.999\,55$$
$$f(-1^\circ) = 3\cos(-1^\circ) = 2.999\,54$$
$$\frac{2.999\,55 - 2.999\,54}{0.01} = 0.001$$

The instantaneous rate of change at $(-1^\circ, 2.99)$ is positive, and so $(0^\circ, 3)$ is a maximum.

e) $f(x) = x^3 - 3x$; $(-1, 2)$

Examine $x = 0$, which is to the right of $(-1, 2)$.

$$f(0.01) = (0.01)^3 - 3(0.01) = -0.029\,999$$
$$f(0) = (0)^3 - 3(0) = 0$$
$$\frac{-0.029\,999 - 0}{0.01} = -2.9999$$

The instantaneous rate of change at $(0, 0)$ is -2.9999, and so $(-1, 2)$ is a maximum.

f) $f(x) = -x^3 + 12x - 1$; $(2, 15)$

Examine $x = 1$, which is to the left of $(2, 15)$.

$$f(1.01) = -(1.01)^3 + 12(1.01) - 1 = 10.0897$$
$$f(1) = -(1)^3 + 12(1) - 1 = 10$$
$$\frac{10.0897 - 10}{0.01} = 8.97$$

The instantaneous rate of change at $(1, 10)$ is 8.97, and so $(2, 15)$ is a maximum.

7. Use a table to inspect several values of $h(t)$.

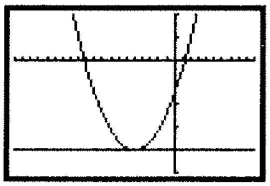
t	h(t)
0	10 000
1	10 074
2	10 116
3	10 126
4	10 104
5	10 050

The height is definitely decreasing after $t = 3$, but for this data the exact maximum cannot be determined. Examine other values of t to help determine the maximum.

t	$h(t)$
2.75	10 126.50
3.25	10 123.50

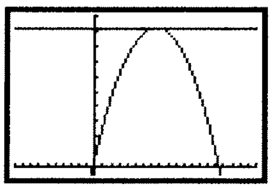
The maximum appears to be pretty close to 2.75. The slopes of tangents for values of t less than about 2.75 would be positive, while slopes of tangents for values of t greater than about 2.75 would be negative.

8. a) i)



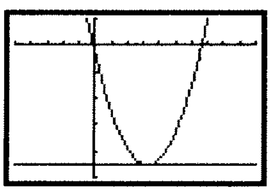
The minimum is at approximately $x = -5$.

ii)



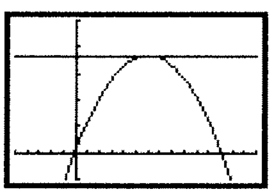
The maximum is at $x = 7.5$.

iii)



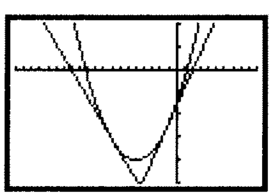
The minimum is at approximately $x = 3.25$.

iv)

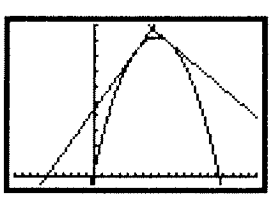


The maximum is at $x = 6$.

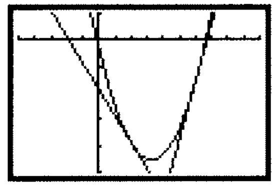
b) i)



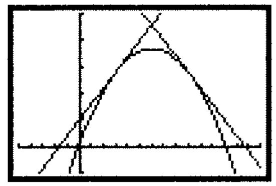
ii)



iii)

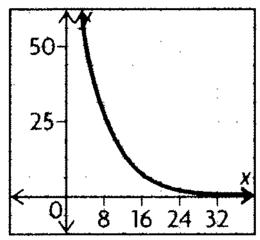


iv)



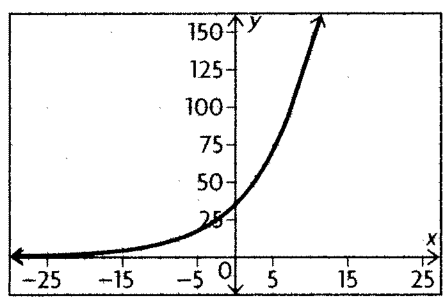
c) Answers may vary. For example, if the sign of the slope of the tangent changed from positive to negative, there was a maximum. If the sign of the slope of the tangent changed from negative to positive, there was a minimum.

9. a) i) Examine the graph of the equation.



The maximum for the interval $0 \leq t \leq 5$ appears to be at $x = 0$ or $(0, 100)$. The minimum appears to occur at $t = 5$ or $(5, 44.4)$. This cannot be verified with the difference quotient because the graph will always be decreasing. This means that the instantaneous rate of change for any point on the graph will always be negative and never be zero.

ii) Examine the graph of the function.



b) The minimum appears to be at $x = 0$ or $(0, 35)$ and the maximum at $x = 10$ or $(10, 141.6)$. This cannot be verified with the difference quotient because the graph will always be increasing. This means that the instantaneous rate of change for any point on the graph will always be positive and never be zero.

10. Answers may vary. For example, examine points on either side of $t = 0.5$ s to make sure that the diver's height is increasing before the point and decreasing afterwards.

$$h(0.49) = -5(0.49)^2 + 5(0.49) + 10 = 11.2495$$

$$h(0.5) = -5(0.5)^2 + 5(0.5) + 10 = 11.25$$

$$\frac{11.25 - 11.2495}{0.01} = 0.05$$

The slope to the right of the point is positive.

$$h(0.51) = -5(0.51)^2 + 5(0.51) + 10 = 11.2495$$

$$h(0.5) = 11.25$$

$$\frac{11.2495 - 11.25}{0.01} = -0.05$$

The function is increasing up to 0.5 s and decreasing after 0.5 s—the point is a maximum.

11. Answers may vary. For example, yes, this observation is correct. The slope of the tangent at 1.5 s is 0.

The slopes of the tangents between 1 s and 1.5 s are negative, and the slopes of the tangent lines between 1.5 s and 2 s are positive. So, the minimum of the function occurs at 1.5 s.

12. Answers may vary. For example, estimate the slope of the tangent line to the curve when $x = 5$ by writing an equation for the slope of an secant line on the graph if $R(x)$. If the slope of the tangent is 0, this will confirm there may be a maximum at $x = 5$. If the slopes of tangent lines to the left are positive and the slopes of tangent lines to the right are negative, this will confirm that a maximum occurs at $x = 5$.

13. Answers may vary. For example, because $\sin 90°$ gives a maximum value of 1, I know that a maximum occurs when $(k(x - d)) = 90°$. Solving this equation for x will tell me what types of x-values will give a maxim. For example, when $k = 2$ and $d = 3$,

$$(2(x - 3°)) = 90°$$

$$(x - 3°) = 45°$$

$$x = 48°$$

14. Myra is plotting (instantaneous) velocity versus time. The rates of change Myra calculates represent acceleration. When Myra's graph is increasing, the car is accelerating. When Myra's graph is decreasing, the car is decelerating. When Myra's graph is constant, the velocity of the car is constant; the car is neither accelerating nor decelerating.

15. Choose a method and determine the instantaneous rates of change for the points given. Use tables to examine the relationship between x and the instantaneous rate of change at x.

x	Rate of Change
−2	−4
−1	−2
2	4
3	6

The instantaneous rate of change appears to be 2 times the x-coordinate or $2x$. Now use a table to examine the relationship between the points given and their instantaneous rates of change for the function $f(x) = x^3$.

x	Rate of Change
−2	12
−1	3
2	12
3	27

The instantaneous rate of change appears to be 3 times the square of the x-coordinate or $3x^2$.

Chapter Review, pp. 116–117

1. a) Examine the rate of change between each interval. If the rate of change is the same for each interval, then the data follows a linear relation.

$$\frac{297.50 - 437.50}{17 - 25} = 17.5$$

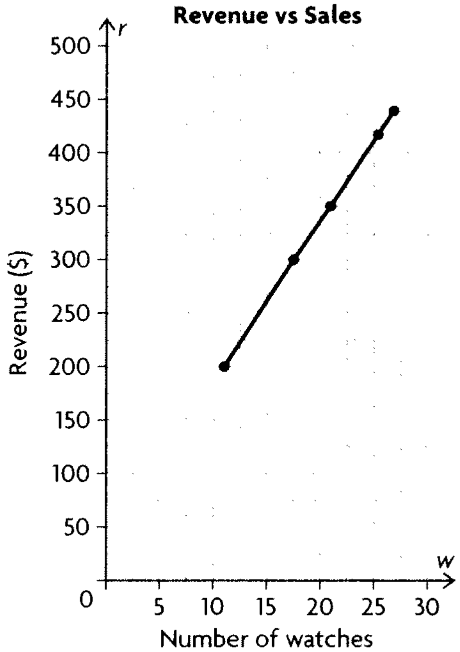
$$\frac{350.00 - 297.50}{20 - 17} = 17.5$$

$$\frac{210.00 - 350.00}{12 - 20} = 17.5$$

$$\frac{420.00 - 210.00}{24 - 12} = 17.5$$

The slope between each interval is the same, and so the relation is linear.

b) Answers may vary. For example:



The graph appears to be linear, and so it would appear that my hypothesis is correct.

c) The average rate of change from $w = 20$ to $w = 25$.

$$\frac{437.50 - 350.00}{25 - 20} = \$17.50 \text{ per watch}$$

d) The cost of one watch is \$17.50; this is the slope of the line on the graph.

2. a) Calculate the average rate of change for the interval [0, 4]. The second point is (4, 7); the first is (0, 1).

$$\frac{7 - 1}{4 - 0} = 1.5 \text{ m/s}$$

b) Calculate the average rate of change for the interval [4, 8]. The second point is (8, 1). The first point is (4, 7).

$$\frac{1 - 7}{8 - 4} = -1.5 \text{ m/s}$$

c) The time intervals have the same length. The amount of change is the same, but with opposite signs for the two intervals. So the rates of change are the same for the two intervals, but with opposite signs.

3. a) The company spends \$2500 per month in expenses—this can be represented by $2500m$. The initial expenses were 10 000. The whole equation is $E = 2500m + 10\,000$.

b) Find the expenses for $m = 6$ and $m = 3$.

$$2500(6) + 10\,000 = 25\,000$$
$$2500(3) + 10\,000 = 17\,500$$
$$\frac{25\,000 - 17\,500}{6 - 3} = 2500$$

The average rate of change is \$2500 per month.

c) No, the equation that represents this situation is linear, and the rate of change over time for a linear equation is constant.

4. a) Answers may vary. For example: Because the unit of the equation is years, do not choose $3 \le t \le 4$ and $4 \le t \le 5$. A better choice would be $3.75 \le t \le 4.0$ and $4.0 \le t \le 4.25$.

b) Answers may vary. For example, the equation is $V(t) = 2500(1.15)^t$. Find $V(4.0)$ and $V(4.25)$.

$$V(4.0) = 2500(1.15)^{4.0} = 4372.515\,625$$
$$V(4.25) = 2500(1.15)^{4.25} = 4527.993\,869$$
$$\frac{4527.993\,869 - 4372.515\,625}{4.25 - 4.0} = 621.912\,976$$
$$V(3.75) = 2500(1.15)^{3.75} = 4222.376\,055$$
$$V(4.0) = 4372.515\,625$$
$$\frac{4372.515\,625 - 4222.376\,055}{4.0 - 3.75} = 600.558\,280$$
$$\frac{600.558\,280 + 621.912\,976}{2} = 611.24$$

5. a) Answers may vary. For example, squeezing the interval.

b) Squeezing the interval will be a good method. Use the interval $11.99 \le t \le 12.01$. The equation is $y = 2\sin(120°t)$.

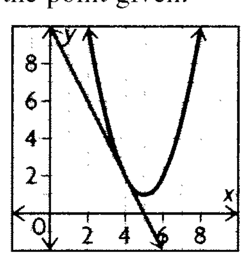
$$2\sin(120°(11.99)) = -0.0419$$
$$2\sin(120°(12.01)) = 0.0419$$
$$\frac{0.0419 - (-0.0419)}{12.01 - 11.99} = 4.19 \text{ cm/s}$$

Now use the interval $11.999 \le t \le 12.001$.

$$2\sin(120°(11.999)) = -0.004\,19$$
$$2\sin(120°(12.001)) = 0.004\,19$$
$$\frac{0.004\,19 - (-0.004\,19)}{12.001 - 11.999} = 4.19 \text{ cm/s}$$

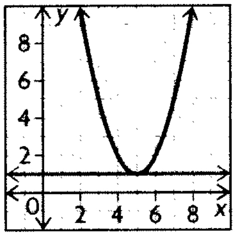
6. For each point, draw a line tangent to the graph at the point given.

a)



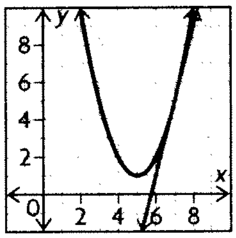
The slope of the line appears to be −2.

b)



The slope of the line appears to be 0.

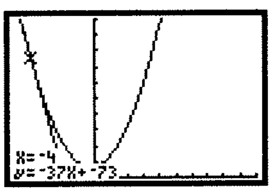
c)



The slope of the line appears to be 4.

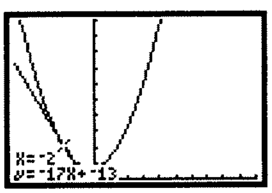
7. Graph the original equation. Find the corresponding y for each value of x given. Use this information to draw a tangent line to the original graph with a graphing calculator.

a)



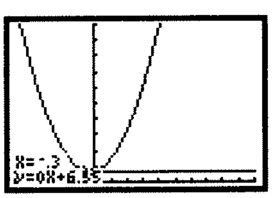
The slope of the line, and therefore the instantaneous rate of change at $x = -4$, is -37.

b)



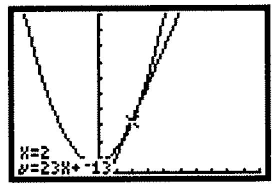
The slope of the line, and therefore the instantaneous rate of change at $x = -2$, is -17.

c)



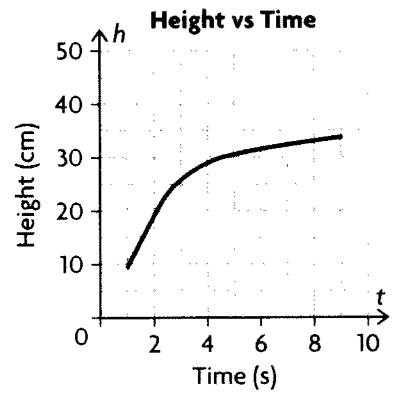
The slope of the line, and therefore the instantaneous rate of change at $x = -0.3$, is 0.

d)

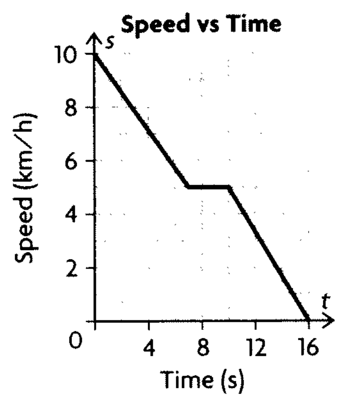


The slope of the line, and therefore the instantaneous rate of change at $x = 2$, is 23.

8.



9. a) Answers may vary. For example:



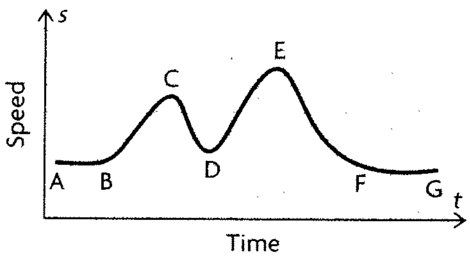
b) Find the average rate of change in the bicycle rider's speed on the interval $0 \leq t \leq 7$. The speed at $t = 0$ was 10 km/h. The speed at $t = 7$ was 5 km/h. The average rate of change in speed is $\frac{5-10}{7-0} = -\frac{5}{7}$ km/h/s.

c) From $(7, 5)$ to $\left(12, \frac{10}{3}\right)$, the average rate of change of speed is $-\frac{1}{3}$ km/h/s.

d) The speed is decreasing at a constant rate from $t = 10$ to $t = 16$. So find the average rate of change on any interval between those two numbers and it will be the same as the instantaneous rate of change at $t = 12$.

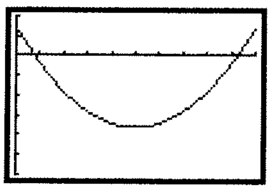
$$\frac{0-5}{16-10} = -\frac{5}{6}\text{ km/h/s}$$

10. The roller coaster moves at a slow steady speed between A and B. At B it begins to accelerate as it moves down to C. Going uphill from C to D it decelerates. At D it starts to move down and accelerates to E, where the speed starts to decrease until, where it maintains a slower speed to G, the end of the track.



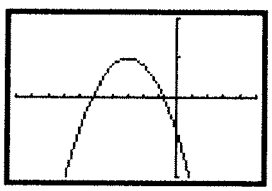
11. Graph each function using a graphing calculator to determine whether the point given is a maximum or a minimum.

a)



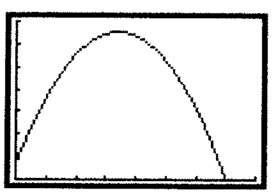
The graph shows that $(5, -18)$ is a minimum.

b)



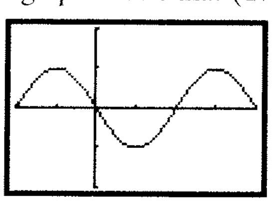
The graph shows that $(-3, 5)$ is a maximum.

c)



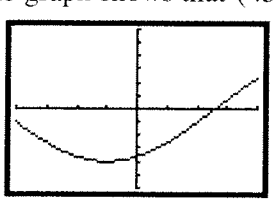
The graph shows that $(17, 653)$ is a maximum.

d)



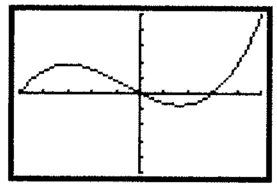
The graph shows that $(45°, -1)$ is a minimum.

e)



The graph shows that $(-25°, -4)$ is a minimum.

f)



The graph shows that $\left(-3, \frac{9}{5}\right)$ is a maximum.

12. a)

i) $f(x) = x^2 - 30x$

$$\begin{aligned} f(2+h) &= (2+h)^2 - 30(2+h) \\ &= 2^2 + 2(2)h + h^2 - 30(2) - 30h \\ &= -56 - 26h + h^2 \\ f(2) &= (2)^2 - 30(2) \\ &= -56 \end{aligned}$$

$$\frac{-56 - 26h + h^2 - (-56)}{2 + h - 2} = h - 26$$

The slope is $m = h - 26$.

ii) $g(x) = -4x^2 - 56x + 16; a = -1$

$$\begin{aligned} g(-1+h) &= -4(-1+h)^2 - 56(-1+h) + 16 \\ &= -4(1 - 2h + h^2) + 56 - 56h + 16 \\ &= -4 + 8h - 4h^2 + 56 - 56h + 16 \\ &= -4h^2 - 48h + 68 \\ g(-1) &= -4(-1)^2 - 56(-1) + 16 \\ &= -4 + 56 + 16 \\ &= 68 \end{aligned}$$

$$\frac{4h^2 - 48h + 68 - 68}{-1 + h - (-1)} = -4h - 48$$

b) For each of the points given, the value of h would be equal to zero. Substitute 0 in for h to find the instantaneous rate of change for each point.

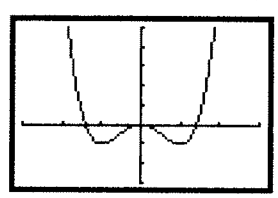
i) $m = 0 - 26 = -26$

ii) $m = -4(0) - 48 = -48$

13. a) To the left of a maximum, the instantaneous rates of change are positive. To the right, the instantaneous rates of change are negative.

b) To the left of a minimum, the instantaneous rates of change are negative. To the right, the instantaneous rates of change are positive.

14. a)



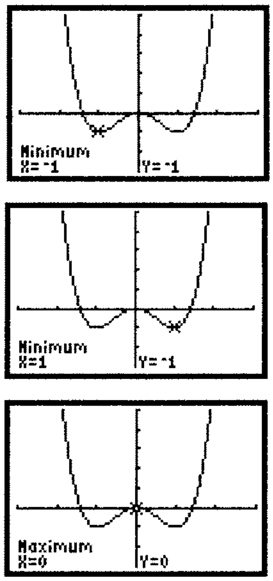
b) minimum: $x = -1, x = 1$

maximum: $x = 0$

c) The slopes of tangent lines for points to the left of a minimum will be negative, while the slopes of tangent lines for points to the right of a minimum will be positive.

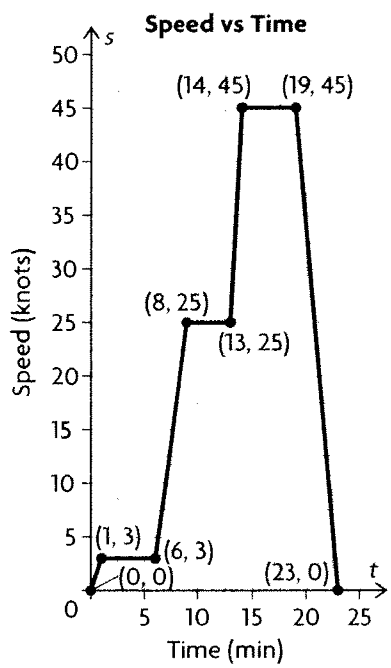
The slopes of tangent lines for points to the left of a maximum will be positive, while the slopes of tangent lines for points to the right of a minimum will be negative.

d)



Chapter Self-Test, p. 118

1. a)



b) At $t = 8$ s the speed is approximately 25 knots. At $t = 6$ the speed is approximately 3 knots.

$$\frac{25 - 3}{8 - 6} = 11 \text{ kn/min}$$

At $t = 13$ the boats speed is 25 knots.

$$\frac{25 - 25}{13 - 8} = 0 \text{ kn/min}$$

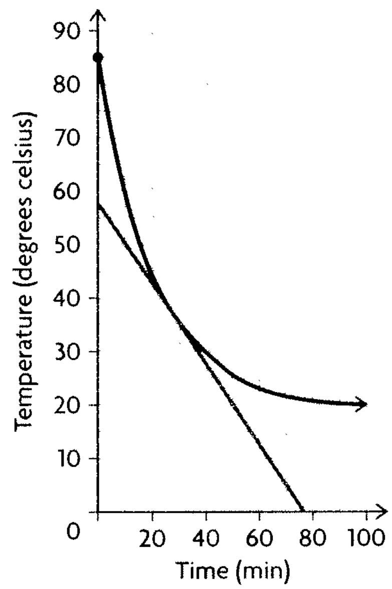
The two different average rates of change indicate that the boat was increasing its speed from $t = 6$ to $t = 8$ at a rate of 11 kn/min and moving at a constant speed from $t = 8$ to $t = 13$.

c) Because the rate of change is constant over the interval, the instantaneous rate of change at $t = 7$ would be the same as it was over the interval, $6 \leq t \leq 8$, 11 kn/min.

2. a) The slope of the secant line between (5, 70) and (50, 25) would be $\frac{25 - 70}{50 - 5} = -1$.

b) The hot cocoa is cooling by 1 °C/min on average.

c) Examine the graph to and draw a line tangent to the graph at the point (30, 35).



The slope of the tangent line is -0.75.

d) The hot cocoa is cooling by 0.75 °C/min after 30 min.

e) The rate of decrease decreases over the interval, until it is nearly 0 and constant.

3. a) Calculate both $P(10)$ and $P(8)$.

$$P(10) = -5(10)^2 + 400(10) - 2550$$
$$= 950$$
$$P(8) = -5(8)^2 + 400(8) - 2550$$
$$= 330$$
$$\frac{950 - 330}{10 - 8} = 310$$

The average rate of change is \$310 per dollar spent.

b) Use the different quotient to estimate the instantaneous rate of change.

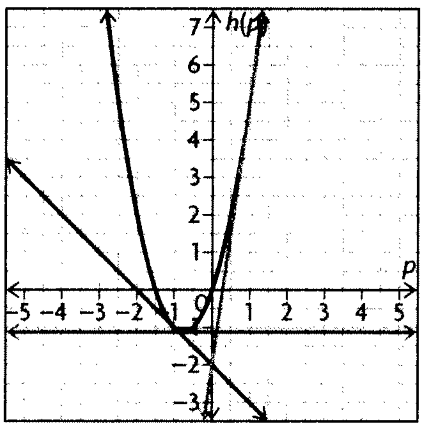
$$P(50.01) = -5(50.01)^2 + 400(50.01) - 2550$$
$$= 4948.9995$$

$$P(50) = -5(50)^2 + 400(50) - 2550$$
$$= 4950$$
$$\frac{4948.9995 - 4950}{0.01} = -100.05$$

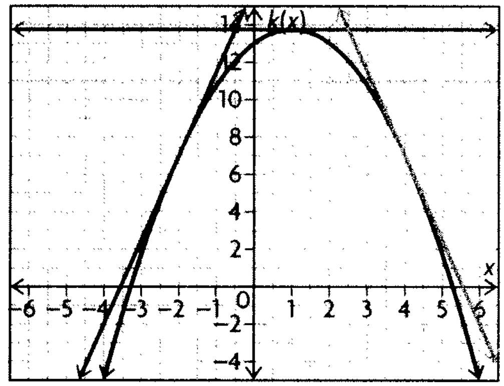
The instantaneous rate of change is approximately $-\$100$ per dollar spent.

c) The positive sign for part a) means that the company is increasing its profit when it spends between \$8000 and \$10 000 on advertising. The negative sign means that the company's profit is decreasing when it spends \$50 000 on advertising.

4. Graph each function and approximate the tangent line at each of the given points. Estimate the instantaneous rate of change at each point given by determining the slope of the tangent line at the given point.

a) 

The instantaneous rate of change when $p = -1$ is -1.
The instantaneous rate of change when $p = -0.75$ is 0. The point is a minimum.
The instantaneous rate of change when $p = 1$ is 7.

b) 

The instantaneous rate of change when $x = -2$ is 4.5.
The instantaneous rate of change when $x = 4$ is -4.5.
The instantaneous rate of change when $x = 1$ is 0. This point is a maximum.